RESEARCH STATEMENT

SIÂN FRYER

1. Introduction

My research area is in noncommutative algebra, in particular quantum algebras from both a ring-theoretic point of view (e.g. their prime and primitive spectrum) and as deformations of commutative algebras. This is currently an extremely active area of research and has links to many other areas of pure mathematics, for example total nonnegativity and algebraic combinatorics, quantum groups and quantum function algebras, and cluster algebras.

Following Dixmier's philosophy, we should study quantum algebras by first classifying their irreducible representations, or if this turns out to be an impossibly large problem we should at least classify the primitive ideals of the algebra: the primitives are by definition the ideals which annihilate a simple module, and the simple modules of an algebra are equivalent to its irreducible representations. One primitive ideal may annihilate many different simple modules, so this reduces the scope of the problem to something more manageable.

For many types of noncommutative algebra, the key technique used to study the prime and primitive ideals is $\mathcal{H}$-stratification: essentially, we take a noncommutative algebra A with a rational action by a torus $\mathcal{H}$, identify and study the prime ideals which are invariant under the action of $\mathcal{H}$ (the "$\mathcal{H}$-primes"), and use this information to break up the prime spectrum $spec(A)$ into well-behaved strata for study. In particular, this is turns out to be a very effective way of identifying and classifying the primitive ideals of A: they are exactly those prime ideals which are maximal in their stratum.

One key area in the study of $\mathcal{H}$-primes and $\mathcal{H}$-stratification is the work of Joseph, Hodges-Levasseur, and Yakimov, which uses techniques from representation theory to construct quantum algebras by analogy to the classical commutative picture. (For those familiar with this language: they are subalgebras of the restricted duals of quantized enveloping algebras of Lie algebras. The more algebraic or combinatorics-minded reader should not be put off by this terminology, however, as many of these algebras also have a nice presentation in terms of generators and relations!) This viewpoint allows us to apply results from quantum groups and representation theory to study the prime spectra of a wide class of noncommutative algebras, as detailed in [24]. I discuss this further in §2 below.

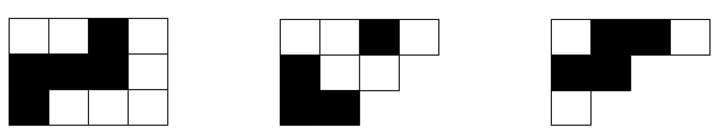

Figure 1. Examples of Cauchon diagrams (also called Le diagrams).

In addition to their usefulness in stratifying $spec(A)$, $\mathcal{H}$-primes are also interesting objects of study in their own right, both from a ring-theoretic and a combinatorial point of view. The $\mathcal{H}$-primes in the algebra of quantum matrices $\mathcal{O}_q(M_{m,n})$ (intuitively: the noncommutative analog of the classical coordinate ring of $m \times n$ matrices) are indexed by *Cauchon diagrams*:

collections of black and white boxes such as those in Figure 1. However, diagrams with exactly the same definition appeared independently under the name *Le diagrams* in Postnikov's work on total nonnegativity in $m \times n$ real matrices and Grassmannians [21], which opens up a whole new world of possible techniques and interesting interactions between algebra and combinatorics.

The study of total nonnegativity in real matrices and in the Grassmannian is described in more detail in §3, but the key idea is as follows: an $m \times n$ matrix with real entries is called *totally nonnegative* if all of its minors are positive or zero. The space of totally nonnegative real matrices decomposes naturally into cells, which are defined by specifying which minors are zero in a given cell; many such cells will be empty, but the non-empty ones correspond exactly to $\mathcal{H}$-primes in $\mathcal{O}_q(M_{m,n})$.

The connection between these two areas turns out to be very deep: in particular, a list of minors defines a non-empty cell if and only if every minor on the list (and no others) are contained in the corresponding $\mathcal{H}$-prime [10, Theorem 4.2]. Problems such as finding generating sets for $\mathcal{H}$-primes can therefore be phrased in combinatorial terms, and conversely questions on (for example) identifying which lists of minors give non-empty cells can be answered using $\mathcal{H}$-stratification techniques.

Finally, the study of $\mathcal{H}$-primes also has unexpected real world applications: Kodama and Williams [13] have shown that totally nonnegative cells can be used to study solutions of the KP equation, which models the behaviour and interaction of shallow water waves. This in turn has important uses in the study and prediction of tsunami waves, for example.

This research statement is arranged as follows. In §2 below, I briefly outline some of the theory of $\mathcal{H}$-stratification and how this relates to a major open conjecture on the structure of *spec*(A) for quantum algebras A. In [8] I showed that this conjecture has a positive answer for the quantized coordinate ring of SL_3, and I am now working on extending these techniques to a wider class of algebras in collaboration with Milen Yakimov. In §2.3, I list several questions for future study and outline how I intend to solve them.

These questions motivate the theme of §3, which considers the same problems from the viewpoint of algebraic combinatorics. I discuss the connection between $\mathcal{H}$-primes and total nonnegativity in more detail, and show that the algebraic problems considered in §2 motivate questions which are also interesting from a purely combinatorial point of view (e.g. as in my recent preprint [7]). This section also ends with several directions for further research and some specific questions that I intend to work on.

Finally, in §4 I indicate another possible research direction for future work, which focuses on the many connections between the topics described above and the study of cluster algebras and cluster categories.

2. $\mathcal{H}$-stratification in quantum algebras

2.1. **Background.** Intuitively, quantum algebras are deformations of classical coordinate rings such as $\mathcal{O}(k^n)$ or $\mathcal{O}(M_{m,n})$, with a parameter q such that when we let q tend to 1 we recover the original commutative coordinate ring. This process induces the additional structure of a Poisson bracket on the commutative algebra (often referred to as the "semi-classical limit"), which retains some of the noncommutative behaviour of the quantized multiplication.

The representation theory of a quantum algebra A splits into two very different cases, depending on whether $q \in k^{\times}$ is a root of unity or not; on the Poisson side, the corresponding condition is whether the field k has characteristic p or 0. I am primarily interested in the generic case, since that is where $\mathcal{H}$-stratification techniques are most effective, so we will assume throughout that q is not a root of unity and $char(k) = 0$. Among other nice properties, this implies that all prime ideals of A are completely prime.

When computing with some small examples, for example the quantum plane

$$k_q[x, y] := k\langle x, y\rangle/(xy - qyx),$$

and corresponding Poisson algebra $k[x, y]$ with bracket $\{x, y\} = xy$, one finds that there is a homeomorphism from $spec(k_q[x, y])$ to $pspec(k[x, y])$ (the set of commutative prime ideals which are closed under the Poisson bracket). This and other examples led Goodearl to make the following conjecture.

Conjecture 2.1. [9, Conjecture 9.1] *Let A be a generic single-parameter quantum algebra with semi-classical limit R. Then there should be a homeomorphism between $spec(A)$ and $pspec(R)$, and this should restrict to a homeomorphism between the primitive ideals of A and the Poisson-primitive ideals of R.*

This conjecture has already been answered positively in several cases (e.g. quantum affine spaces [11], odd and even dimension Euclidean spaces [18, 19]). In [24], Yakimov constructed a $\mathcal{H}$-equivariant bijection between the two spaces $spec(A)$ and $pspec(R)$ for a very general class of quantum algebras, but the question of whether this is actually a homeomorphism remains open.

2.2. **My work in this area.** For the case of quantum matrices $\mathcal{O}_q(M_{m,n})$ (and related algebras $\mathcal{O}_q(GL_n)$, $\mathcal{O}_q(SL_n)$, which are the quantized coordinate rings of GL_n and SL_n respectively), Conjecture 2.1 was only known to be true for the case $\mathcal{O}_q(SL_2)$ until very recently; in [8] I proved the following theorem.

Theorem 2.2. *Let k be an algebraically closed field of characteristic 0 and $q \in k^{\times}$ not a root of unity. Give $\mathcal{O}(SL_3)$ and $\mathcal{O}(GL_2)$ the Poisson bracket induced from $\mathcal{O}_q(SL_3)$, $\mathcal{O}_q(GL_2)$ respectively. Then there exist homeomorphisms $spec(\mathcal{O}_q(SL_3)) \longrightarrow pspec(\mathcal{O}(SL_3))$ and $spec(\mathcal{O}_q(GL_2)) \longrightarrow pspec(\mathcal{O}(GL_2))$, and these restrict to homeomorphisms on the primitive/Poisson-primitive spectra as well.*

The reason that this is such a difficult problem in general is as follows. Finding the homeomorphisms predicted by Conjecture 2.1 is essentially equivalent to understanding inclusions of primes in both the quantum algebra and the Poisson algebra; however, while $\mathcal{H}$-stratification gives us an excellent picture of each individual stratum, it tells us nothing about the interactions of different strata. The case $\mathcal{O}_q(SL_2)$ could be solved quite easily since it only has four very small strata, so its inclusions could be computed explicitly; on the other hand, $\mathcal{O}_q(SL_3)$ admits 36 $\mathcal{H}$-primes, and we would need to be able to compare primes from the strata of all possible pairs of $\mathcal{H}$-primes $J \subsetneq K$. This problem grows very quickly: for example, $\mathcal{O}_q(M_{5,5})$ admits 329,462 $\mathcal{H}$-primes.

In order to prove Theorem 2.2 for $\mathcal{O}_q(SL_3)$ and $\mathcal{O}_q(GL_2)$, I made use of an extension to the $\mathcal{H}$-stratification framework proposed by Brown and Goodearl in [3]. A key part of this is the following: if $J \subsetneq K$ are two $\mathcal{H}$-primes in a quantum algebra A, we study the intermediate algebra $Z_{JK} := Z(A/J[\mathcal{E}_{JK}^{-1}])$, where $\mathcal{E}_{JK}$ consists of all regular $\mathcal{H}$-eigenvectors in A/J which are not in K. Intuitively, we are restricting our attention to the sub-poset of the $\mathcal{H}$-primes with minimal element J and maximal element K, and we can use this algebra to keep track of inclusions of primes from the strata associated to J and K.

There are three main difficulties which need to be overcome in order to apply this framework:

(1) $\mathcal{E}_{JK}$ need not be an Ore set in general; Brown and Goodearl give an alternative definition in [3] which agrees with this localization when it exists, and also give conditions for constructing Z_{JK} as a localization in terms of smaller Ore set E_{JK} [3, Lemma 3.9]. However, this does not guarantee that such an Ore set will exist.
(2) Very little is known about the structure of the algebras Z_{JK}; all examples computed so far have been affine, but it is not known if this will hold in general.
(3) In order to apply the framework of [3], one must verify that all prime ideals in the algebra satisfy certain technical conditions; this is currently only known for a few small examples.

In [8], I extended the framework of [3] to the case of Poisson algebras with a $\mathcal{H}$-action, and computed the structure of the algebras Z_{JK} directly for both $\mathcal{O}_q(SL_3)$ and $\mathcal{O}(SL_3)$. This provided enough information about the topological structure of $spec(\mathcal{O}_q(SL_3))$ and $pspec(\mathcal{O}(SL_3))$ to obtain Theorem 2.2 above.

2.3. **Future work.** In joint work with Milen Yakimov (which is currently in preparation), we use the language of quantum groups and representation theory to construct Ore sets E_{JK} with the desired properties for a wide class of quantum algebras. This allows us to simplify the Brown-Goodearl framework by showing that we can always realise the intermediate algebras Z_{JK} as centres of localizations. The following questions now become more tractable.

Question 1. What is the structure of the algebras Z_{JK} in general? In all examples computed so far, they take the form $k[Z_1^{\pm 1}, \ldots, Z_r^{\pm 1}, Z_{r+1}, \ldots, Z_s]$ for some generators Z_i; is this always the case, and can we find a way to describe these central generators explicitly, e.g. by analogy to [1]?

Question 2. Does the Brown-Goodearl conjecture [3, Conjecture 3.11] hold for general quantum algebras, i.e. does the framework described above give a full and accurate picture of the topological space $spec(A)$?

In [3], a sufficient condition is given for the conjecture to hold in a quantum algebra: all prime ideals of A should be generated by normal elements modulo their $\mathcal{H}$-prime. This is a difficult condition to check in general, as the algebra A/J is not necessarily very easy to work with. By realising Z_{JK} as the centre of a localization, however, I have shown that we can give another sufficient condition for the conjecture to hold, which only requires us to understand the algebras $A/J[E_{JK}^{-1}]$ and Z_{JK}. I am therefore hopeful that the answer to Question 2 will follow from the answer to Question 1.

Question 3. Can we give a purely ring-theoretic construction of the Ore sets E_{JK}, or find a way to compute them explicitly?

While the language of quantum groups is extremely powerful, it can be difficult to translate these results into ring-theoretic terms. For example, we might want to compute actual examples of the algebras Z_{JK} in the case of quantum matrices, in order to make use of these results in studying totally nonnegative matrices.

This is where it becomes practical to translate some of these questions into the language of combinatorics. The framework of total nonnegativity (which is described in §3.1 below) turns out to be extremely effective for tackling Question 3, while the algebraic motivation ensures that we end up with questions that are also interesting from a purely combinatorial point of view.

3. Combinatorics and Total Nonnegativity

3.1. **Background.** In order to apply the $\mathcal{H}$-stratification framework described in §2, one of the key requirements is a good understanding of the $\mathcal{H}$-primes themselves: finding computationally feasible generating sets, identifying whether a given $\mathcal{H}$-eigenvector is in the $\mathcal{H}$-prime or not, constructing Ore sets intersecting certain ideals but not others, and describing the structure of the quotients A/J. Many of these questions translate very elegantly to the language of algebraic combinatorics and total nonnegativity, which we now describe.

In the vector space of $m \times n$ real matrices $M_{m,n}(\mathbb{R})$, one can consider the closed subspace of totally nonnegative matrices $M_{m,n}^{tnn}(\mathbb{R})$: matrices in which every minor is positive or zero. This space splits naturally into cells as follows: if $\mathcal{F}$ is a family of minors, a matrix M belongs to the cell $Z_{\mathcal{F}}$ if $\mathcal{F}$ lists exactly the set of minors of M which are zero. These cells are often referred to as *positroid cells*. The concept of total nonnegativity extends naturally to the Grassmannian $Gr(k,n)$, where the corresponding requirement for an element to belong to $Gr(k,n)^{tnn}$ is that all of its Plücker coordinates are nonnegative.

As noted in §1, there is a very close connection between the study of $\mathcal{H}$-primes in quantum matrices and positroid cells in totally nonnegative real matrices; this is formalized in the following theorem of Goodearl, Launois and Lenagan.

Theorem 3.1. [10, Theorem 4.2] *Let $\mathcal{F}$ be a family of minors in the coordinate ring $\mathcal{O}(M_{m,n})$ and let $\mathcal{F}_q$ be the corresponding family of quantum minors in $\mathcal{O}_q(M_{m,n})$. Then the following are equivalent:*

(1) The totally nonnegative cell associated to $\mathcal{F}$ in $M_{m,n}^{tnn}(\mathbb{R})$ is non-empty.
(2) $\mathcal{F}$ is the set of all minors that belong to some Poisson $\mathcal{H}$-prime in $\mathcal{O}(M_{m,n})$.
(3) $\mathcal{F}_q$ is the set of all quantum minors that belong to some $\mathcal{H}$-prime in $\mathcal{O}_q(M_{m,n})$.

Theorem 3.1 allows us to combine techniques from algebra and combinatorics in interesting new ways: for example, the deleting-derivations algorithm developed by Cauchon to study prime ideals in quantum algebras has been applied to the problem of identifying whether a real matrix is nonnegative and which cell it belongs to [14], while in the other direction the graph-theoretic techniques used by Postnikov in [21] have been adapted by Casteels in [4] to answer a long-standing question about whether $\mathcal{H}$-primes are generated by their (quantum) minors.

Understanding either the $\mathcal{H}$-primes or the positroid cells is not just an arbitrary problem: as described in §1, the $\mathcal{H}$-primes of a quantum algebra are the gateway to its representation theory, while the positroid cells have applications to physics in the form of modelling shallow water waves (e.g. [13]).

However, while there are many different techniques which converge at the study of $\mathcal{H}$-primes and nonnegativity: the Weyl group approach of Marsh-Rietsch [17], the Cauchon diagrams used by Cauchon [5] and Launois-Lenagan [14], Casteels' graph-theoretic techniques [4], and

Suho Oh's combinatorial approach [20] to name but a few, what is often missing is a way to translate between them. Each of the techniques listed above have their own strengths and weaknesses, and what is difficult or impossible to prove from one point of view often becomes far easier when translated to another.

3.2. **My work in this area.** In [7], I show that we can combine several of these approaches to obtain the following theorem. For brevity, we just note here that a Grassmann necklace is a combinatorial object associated to a positroid cell, lacunary sequences are an algorithm introduced by Cauchon to construct quantum minors with certain properties directly from a Cauchon diagram, and that every $\mathcal{H}$-prime in $\mathcal{O}_q(M_{m,n})$ can be associated to a permutation in the symmetric group S_{m+n} in a unique way; the interested reader is referred to [7] for the full definitions.

Theorem 3.2. *Let K be a $\mathcal{H}$-prime in $\mathcal{O}_q(M_{m,n})$, and let $v \in S_{m+n}$ be the permutation associated to it. Denote the interval $\{a, a+1, \dots, b\}$ by $[\![a, b]\!]$.*

(1) *The $m+n-1$ quantum minors constructed from the sets $v[\![1, k]\!]$ ($1 \leq k \leq m+n-1$) using the formulas of Yakimov in* [23, §5] *(i.e. the quantum groups approach) can also be constructed directly from the Cauchon diagram of K using a reversed version of the lacunary sequence construction.*

(2) *These $m+n-1$ minors correspond exactly to the Grassmann necklace of the positroid cell associated to K, and hence the multiplicative set generated by these minors has exactly the desired properties for constructing the algebras Z_{JK} from §2 for all $\mathcal{H}$-primes $J \subsetneq K$ in $\mathcal{O}_q(M_{m,n})$.*

From Theorem 3.2 we also obtain a simple formula for the Grassmann necklace of the positroid cell associated to K in terms of the permutation v; while several other formulas exist, they are often far harder to compute with directly. This justifies our claim above that we can also obtain interesting combinatorial results from an algebraically-motivated question.

3.3. **Future work.** Using Theorem 3.2 as a blueprint, this motivates the following questions.

Question 4. (a) How we use representation theory (which has been very successful on the quantum algebra side) to study positroids in the Grassmannian and more general flag varieties?

(b) Conversely, can we use combinatorial techniques to obtain a more concrete description of existing results phrased in terms of quantum groups, such as the work of Yakimov and myself on constructing denominator sets for the algebras Z_{JK} (as described in §2.2)?

In the case of quantum matrices, the answers to Question 4 are given by Theorem 3.2; as a result, I expect that I will be able to extend this to more general quantum algebras in future and that the answers will be of wider interest than just the specific problem described in §2.

So far we have only looked at decompositions of the totally nonnegative part of the Grassmannian; however, the positroid decomposition in $Gr(k, n)^{tnn}$ also induces a decomposition of the full Grassmannian which has interesting properties of its own, which we now describe.

In $M_{m,n}^{tnn}(\mathbb{R})$ and $Gr(k, n)^{tnn}$, there is no natural refinement of the positroid decomposition since the state of every minor (zero or positive) is already specified. Some of these minors are consequences of others, however: for example, if $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in M_{2,2}^{tnn}(\mathbb{R})$ and we specify that $a, b, c, (ad - bc) > 0$, then clearly we must have $d > 0$ as well.

In the full ring $M_{2,2}(\mathbb{R})$, on the other hand, the conditions $a, b, c, (ad - bc) \neq 0$ do not determine whether d is zero or non-zero. This is an example of splitting a positroid cell into two *Deodhar components*. We will not define general Deodhar components here, but instead just state some useful facts about them: they form a decomposition of the Grassmannian $Gr(k, n)$ which refines the positroid decomposition, each component has the form $(k^{\times})^a \times k^b$ for some $a, b \geq 0$, and the intersection of a Deodhar component with $Gr(k, n)^{tnn}$ is exactly one positroid cell of $Gr(k, n)^{tnn}$ if $b = 0$ and empty otherwise. (See, e.g., [22] for more details.)

Drawing intuition from the close links between positroid cells in the nonnegative Grassmannian and individual $\mathcal{H}$-strata in quantum algebras (e.g. [2, 14]), I plan to explore how we can use Deodhar components to study the algebras Z_{JK} described in §2. In particular, the coordinate rings of Deodhar components have exactly the structure predicted for the Z_{JK} in Question 1.

To finish this section, I describe two combinatorial questions involving Deodhar components which I plan to work on in the near future.

Question 5. Deodhar components are indexed by Go diagrams, which are a generalisation of Le/Cauchon diagrams, but Go diagrams are currently defined only in terms of the corresponding Deodhar component. Is there a purely diagram-based characterization of Go diagrams, and can any of the techniques developed for use on Le/Cauchon diagrams (e.g. the lacunary sequences in [5, 14], or the "chain rooted at box (x, y)" in [20]) be generalized to Go diagrams?

Question 6. Can the intuition behind Marsh and Rietsch's work [17], i.e. Deodhar components as the set of Borel subgroups which can be reached from a given starting point under the action of certain elements of the Weyl group, be transferred to the quantum algebra side? Can this be used to "step" between two $\mathcal{H}$-primes $J \subset K$ while preserving information about inclusions of primes?

4. Cluster Algebras

The study of quantum cluster algebra structures on quantum algebras and quantum Richardson varieties (which come from the quantized coordinate rings of unions of Deodhar components) is currently an extremely active area of research, e.g. Goodearl-Yakimov [12], Lenagan-Yakimov [16], Leclerc [15].

In particular, Lenagan and Yakimov explore in [16] the link between cluster structures on quantum Schubert cells and certain sets of quantum minors obtained from Cauchon's deleting derivations algorithm, which for quantum matrices can be read off directly from the Cauchon diagram. Again, this combines techniques from quantum groups and ring theory to obtain results which would not have been possible with only one viewpoint or the other, and I look forward to exploring how this can be extended in future.

Further, one of the key techniques I made use of in [7] also appears in the PhD thesis of Nicholas Chevalier [6], where they are used to study cluster category structures related to partial flag varieties and in particular the Grassmannian. This is a research direction about which I currently know very little but would be very interested in exploring further.

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