One Division Ring To Rule Them All

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1. What is Algebra?

In the context of pure mathematics, algebra is the study of the *structure* of mathematical objects. Instead of looking specifically at numbers or vectors or functions, we just think about sets of elements which follow certain rules and see what propeties we can deduce from those rules alone. The advantage of this approach is that anything we prove will be true in *all* structures with these properties, not just the ones we're already familiar with.

All of the numbers we're used to dealing with satisfy a number of rules which we generally take for granted. Four important examples are:

- **Associative**: $a + (b + c) = (a + b) + c$ and $a(bc) = (ab)c$ for all a, b, c.
- **Distributive**: $a(b + c) = ab + ac$ and $(a + b)c = ac + bc$ for all a, b, c.
- **Identities**: There are elements 0 and 1 such that $0 + a = a + 0 = a$ and $1a = a1 = a$ for all a.
- **Additive inverses**: For every a, there is an element $-a$ such that $a + (-a) = 0$.

A set of elements satisfying all of these properties is called a **ring**.

2. To Commute or Not to Commute?

Here's something else we often taken for granted when working with numbers:

- **Commutative**: $a + b = b + a$ and $ab = ba$ for all elements a, b.

In fact, while we can prove that *every* ring must have commutative addition, there are many examples of rings with non-commutative multiplication appearing in mathematics, physics, computer programming and beyond.

One example of a non-commutative ring is $\mathcal{M}_2(\mathbb{R})$ (see "Rings are Everywhere", right); another example is described below.

3. What is a Division Ring?

Rings all have a lot in common, but they also have some differences: for example, it makes sense to *divide* one fraction by another because the answer is another fraction (in other words, we can always write $\frac{a/b}{c/d} = \frac{ad}{bc}$), but we can't always divide one integer by another and get an integer answer.

In maths terms, if we can divide by an element and always get an answer of the same type then we say it "has a multiplicative inverse"; if every non-zero element in the ring has a multiplicative inverse *which is also in that ring*, we call it a **division ring**.

For example, $\mathbb{Z}$ (the ring of all integers, or "whole numbers") is not a division ring since $\frac{1}{2}$ is not an integer, i.e. 2 doesn't have a multiplicative inverse in $\mathbb{Z}$. On the other hand, $\mathbb{Q}$ and $\mathbb{R}$ *are* division rings.

4. Example of a Non-commutative Division Ring

Division rings can also be non-commutative. *Hamilton's quaternions*, which play an important role in many parts of physics, are one such example. To define them, we introduce three new symbols: **i**, **j** and **k**, and declare that they will follow the rule

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1. \qquad (\bigstar)$$

The quaternions (often denoted $\mathbb{H}$) are the set of all elements of the form $a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$, where a, b, c, d are all real numbers. Using the equalities in ($\bigstar$) above, we can work out how **i**, **j** and **k** interact with each other under multiplication; the table to the left highlights their non-commutative behaviour.

×	**1**	**i**	**j**	**k**
1	1	i	j	k
i	i	-1	k	$-j$
j	j	$-k$	-1	i
k	k	j	$-i$	-1

Multiplication table for $\mathbb{H}$.

Rings are Everywhere

These are all examples of rings:

- $\mathbb{Z}$ - the *integers*, the set of all whole numbers (positive, negative and zero).
- $\mathbb{Q}$ - the *rationals*, the set of all fractions of integers.
- $\mathbb{R}$ - the *reals*, all possible decimals including ones with infinite expansions like π.
- $\mathcal{M}_2(\mathbb{R})$ - the set of all 2×2 matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ whose entries are real numbers.

Foundations of Ring Theory

Emmy Noether

The name "ring" is usually attributed to David Hilbert (1862 - 1943), who studied a type of structure he called a *Zahlring* (number ring), but the undisputed founder of modern ring theory was fellow German mathematician **Emmy Noether** (1882 - 1935) pictured above. Noetherian rings (of which division rings are one example) are still a central object of study in mathematics today.

5. New Division Rings from Old

Division rings come in all shapes and sizes, but if we want to find examples of completely new structures it can be difficult to know where to start.

One way to construct new division rings from old is to define a mapping which "moves around" the ring's elements, and focus on the subset of elements which aren't affected. For example, we can define a mapping on $\mathbb{H}$ that replaces **i**, **j**, **k** with $-\mathbf{i}$, $-\mathbf{j}$, $-\mathbf{k}$ in each element $a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$: the elements which are unchanged under this mapping are exactly the real numbers $\mathbb{R}$ and nothing else.

By doing this we've constructed a division ring which is fundamentally different to the one we started with: $\mathbb{R}$ is commutative, while $\mathbb{H}$ is not. This time it was fairly easy to see what the new ring was, but for more complicated examples this won't always be the case.

6. The Problem: Non-commutative Fractions

I focused on one particular division ring, and looked at mappings which only moved the elements around very slightly: this meant my new division ring would be very similar in size to the old one. The size of the new ring was important, since this project was part of a larger effort to find all possible division rings of a certain size.

The elements of this division ring are fractions ab^{-1}: it's the same idea as normal fractions $\frac{a}{b}$ in $\mathbb{Q}$, but instead of being numbers, a and b come from a ring with non-commutative multiplication.

The effect of non-commutativity on the structure of the division ring is huge: even basic operations like multiplication are now much harder to work out, since

$$ab^{-1} \neq b^{-1}a, \quad \text{and} \quad ab^{-1}cd^{-1} \neq acb^{-1}d^{-1}.$$

My first challenge was to find a way around this. I developed a method of "breaking up" a fraction into infinitely many small pieces, similar to the decimal expansion of a normal fraction, e.g.

$$\frac{1}{3} = 0.333\cdots = \frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \ldots$$

I could then program a computer to work with (finitely many of) these fraction pieces, and hence approximate the answer to various calculations. This allowed me to construct enough fractions to describe the structure of the new ring.

Unfortunately, the "nicest" fraction produced by the computer was still more than 10 pages long, so there was also a lot of human intuition and ingenuity required to translate these answers into something useful!

7. Conclusions: One Ring to Rule Them All?

Very little is known about division rings of this type: the computational difficulties make them very difficult to study. One consequence of my work was to show that they are even more alien than we realised, as I proved that we could define some strange new mappings on them which simply can't exist on commutative rings.

I also succeeded in my primary aim, which was to describe the structure of the newly constructed rings: I was able to prove that in each case the rings were identical copies of the original ring.

Or to put it another way: in this case there genuinely *is* only one division ring to rule them all!

References

Fryer, S. *The q-division ring and its fixed rings*. J. Algebra 402 (2014), 358-378.